PERGAMON

International Journal of Multiphase Flow 28 (2002) 1687–1696

International Journal of Multiphase Flow

www.elsevier.com/locate/ijmulflow

Lateral migration in sheared suspensions: a case study of the 'diffusion' model

S. Haber [a,*], H. Brenner [b]

[a] Faculty of Mechanical Engineering, Department of Mechanical Engineering, Technion—Israel Institute of Technology, Haifa 32000, Israel

[b] Department of Chemical Engineering, MIT, Cambridge, MA 02139-4307, USA

Received 20 September 2000; received in revised form 3 June 2002

Abstract

The 'diffusion' model for lateral migration flux has successfully been used to describe shear-induced particle migration in concentrated suspensions of non-Brownian particles subject to simple shear flows. Subsequent analyses, which included the linear momentum equation, attempted to embed this model into a more comprehensive framework that included general inhomogeneous shear flows and their concomitant pressure gradients. Based upon the latter, more general framework, the present paper presents a case study of a simple suspension flow that leads to a prediction that contradicts Darcy's law. The explicit example considered involves the steady, radial, low Reynolds number flow of a concentrated suspension of neutrally buoyant, non-Brownian spheres permanently confined within the annular space between two concentric spherical shells, each shell being permeable only to the interstitial fluid. As such, the annular domain contains a time-independent dispersion of spherical particles permanently confined within its boundaries, while the interstitial fluid flows past these fixed-in-space 'suspended' particles. The foregoing general model for this suspension flow consists of: (i) local mass conservation equations for both the fluid and suspended particles phases; (ii) an overall mass conservation equation for the confined particulate phase; (iii) the constitutive equation for the so-called 'diffusive' particle flux; and (iv) the linear momentum equation governing the local mass-average velocity. The analysis which follows examines the plausibility of the resulting predictions of the radial particle distribution within the annular space, as well as the direction of the radial pressure gradient (relative to the direction of the interstitial flow) required to maintain the steady fluid motion. Although the accepted radial migration/suspension flow equations predict a plausible spatial particle distribution in the annular region, they nevertheless predict the local pressure gradient to be invariant to the direction of the interstitial flow, and to depend upon the viscosity gradient—both conclusions being in conflict with Darcy's law for flow through a 'stationary' bed of particles, which would be expected to apply to our example problem. This predictive failure of the foregoing diffusion model suggests the need

* Corresponding author.
E-mail address: mersh01@tx.technion.ac.il (S. Haber).

PII: S0301-9322(02)00044-7

for a significant modification of the suspension-scale momentum equation, at least in circumstances where large particle concentration gradients exist.

Keywords: Suspension flow; Lateral migration of particles; Porous media

1. Introduction

Leighton and Acrivos (1986) observed the existence of cross-streamline migration of suspended particles in concentrated suspensions of spheres subjected to a macroscopically inhomogeneous, albeit unidirectional, shear flow—a phenomenon that was later termed 'shear-induced diffusion' (Leighton and Acrivos, 1987). According to this phenomenon, in the presence of gradients in the suspension-scale shear rate the particulate and fluid phases (species) move with different velocities, resulting in a net migration of the suspended particles relative to the surrounding fluid. Several computational models (e.g., Stokesian dynamics, cf. Bossis and Brady, 1988; Nott and Brady, 1994) as well as various phenomenological models have been proposed in an attempt to quantify particle migration phenomena in suspension-flow situations other than inhomogeneous unidirectional shear flows. A widely accepted model of this lateral migration phenomenon attributes the disparity existing between the respective phase velocities and the local velocity $\mathbf{v}$ of the suspension as a whole to a 'diffusional' flux, the latter animated by the existence of both suspension-scale shear and particle concentration gradients. Unidirectional or quasi-unidirectional flows have been extensively investigated in the past (Schaflinger et al., 1990; Nir and Acrivos, 1990; Philips et al., 1992; Koh et al., 1994; Acrivos et al., 1993). Zhang and Acrivos (1994) addressed the viscous resuspension of heavy (i.e., non-neutrally buoyant) particles in fully developed laminar pipe flow, extending the unidirectional model to non-one-dimensional flows. Miskin et al. (1999) investigated the stability of a two-dimensional resuspension flow. The initial treatment of Leighton and Acrivos (1987) employed a diffusional model to phenomenologically quantify their experimental unidirectional flow observations. Their interpretation was entirely kinematical in nature. In particular, an adjoint momentum equation was neither required nor addressed. In an attempt to extend the analysis to other flow configurations, subsequent researchers adopted an invariant tensorial form for the diffusional aspect of the process, in addition to including a momentum equation for the mass-averaged velocity, as required to achieve closure (Schaflinger et al., 1990; Zhang and Acrivos, 1994; Miskin et al., 1999).

The following comprehensive set of equations have been adopted by many researchers to model lateral migration phenomena in inertia-free suspensions (see Eqs. (10)–(13) of Zhang and Acrivos (1994), and Eqs. (4), (5), (8) of Miskin et al. (1999)):

(i) momentum:

$$\nabla p = \mu \nabla^2 \mathbf{v} + 2(\nabla \mu) \cdot \mathbf{S}, \tag{1a}$$

(ii) overall mass or volume conservation:

$$\nabla \cdot \mathbf{v} = 0, \tag{1b}$$

(iii) particle conservation:

$$\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{n}_\phi = 0, \quad \mathbf{n}_\phi = \phi\mathbf{v} + \mathbf{j}_\phi, \quad \mathbf{j}_\phi = -D_\phi\nabla\phi - D_\gamma\nabla\dot{\gamma}. \tag{1c}$$

The first member of the above trio represents the low Reynolds number linear momentum equation for a Newtonian continuum possessing a locally variable viscosity, wherein $\mathbf{v}$ is the mass-average velocity, p the pressure, μ the local suspension viscosity, and $\mathbf{S} = (1/2)(\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{T}})$ the shear-rate dyadic—each measured at the suspension scale. The second equation, representing the overall continuity equation, incorporates the fact that the particles and fluid possess the same densities, so that the mass density ρ is uniform throughout the suspension for all time. The third equation is a mass (or, equivalently, volumetric) conservation equation for the particulate phase, with ϕ the local volumetric concentration of suspended particles, $\mathbf{n}_\phi$ the total volumetric particle flux—consisting of a convective volumetric particle flux $\phi\mathbf{v}$ and a 'diffusive' volumetric particle flux $\mathbf{j}_\phi$. The above constitutive equation for the diffusive contribution, due to Leighton and Acrivos (1987), embodies both concentration and shear diffusivities, while supposing Brownian motion to be absent. Here, $\dot{\gamma} = (2\mathbf{S}:\mathbf{S})^{1/2}$ is a *positive* scalar representing a characteristic local shear rate based on the second invariant of the shear-rate dyadic, whereas D_ϕ and D_γ are, respectively, the concentration and shear-induced diffusion coefficients.

The functional forms of the ϕ-dependencies of the viscosity and diffusivity coefficients varies slightly from author to author (e.g., Krieger, 1972; Philips et al., 1992; Zhang and Acrivos, 1994). We adopt here the respective formulations suggested by Zhang and Acrivos (1994):

$$\mu = \mu_0\left[1 + \frac{1.5\phi}{1 - (\phi/\phi^*)}\right]^2, \tag{2a}$$

$$D_\phi = a^2\dot{\gamma}\phi^2 f(\phi,\phi^*), \quad f(\phi,\phi^*) = \left[\frac{1}{3} + \frac{1}{6}\exp(8.8\phi) + \frac{0.6}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\phi}\right], \tag{2b}$$

$$D_\gamma = 0.6a^2\phi^2 \tag{2c}$$

with μ_0 the interstitial fluid viscosity, a the particle radius, and ϕ^* the solid-phase volume fraction above which the suspension can no longer flow (chosen in the subsequent calculations to be 0.58, as in the work of Zhang and Acrivos (1994)). Below, we solve the preceding system of equations for the radial flow case described in Abstract.

2. Radial flow

Let R_1 and R_2, respectively denote the inner and outer radii of the permeable shells. We suppose that the radial flow occurs at a constant volumetric flow rate Q (which will be taken to be positive for outflow and negative for inflow). Obviously, a steady state will eventually be reached, in which the particles will presumably tend to accumulate closer to the outer wall when $Q > 0$ and, conversely, to the inner wall when $Q < 0$. The existence of a steady state requires that no flux of the particulate phase occur at any radial position, since the bounding walls do not permit particle penetration. On the other hand, the interstitial fluid phase will seep between the particles and through the porous boundaries at a steady rate. As such, *relative* motion occurs locally between the particulate and fluid phases, just as in the case of conventional lateral migration phenomena and flow through porous media.

Since the flow rate Q refers the volume-average flow rate of the suspension, the volume- (or equivalent mass-) average radial velocity component $v_r = v_r(r)$ at any point r $(R_1 < r < R_2)$ can easily be derived from the continuity Eq. (1b) alone as

$$v_r = \frac{Q}{4\pi r^2}. \tag{3}$$

Hence, with $\mathbf{i}_r$ a unit vector in the radial direction,

$$\mathbf{S} = \frac{Q}{4\pi r^3}(\mathbf{I} - 3\mathbf{i}_r\mathbf{i}_r), \quad \dot{\gamma} = \frac{\sqrt{3}|Q|}{2\pi r^3}. \tag{4}$$

Requiring that $\phi \equiv \phi(r)$ be time independent as a consequence of the assumed steady state yields a simplified form of Eq. (1c), namely

$$v_r\phi - D_\phi \frac{\mathrm{d}\phi}{\mathrm{d}r} - D_\gamma \frac{\mathrm{d}\dot{\gamma}}{\mathrm{d}r} = 0 \tag{5}$$

expressing the fact that the total flux $\mathbf{n}_\phi$ of the particulate phase vanishes for all r.

Introduction of (2a), (2b), (2c), (3) and (4) into (5) results in the following dimensionless first-order equation for ϕ:

$$f(\phi, \phi^*)\frac{\mathrm{d}\phi}{\mathrm{d}\hat{r}} - \frac{1.8}{\hat{r}} = \frac{(\mathrm{sgn}Q)}{2\sqrt{3}}\frac{\hat{r}}{\phi}, \tag{6}$$

where $\hat{r} = r/a$ is the dimensionless radial coordinate scaled with the particle radius; $\mathrm{sgn}Q = Q/|Q|$ is either $+1$ or -1 according as Q is positive or negative. Both the flow direction and ϕ^* play key roles in the properties of the resulting solution. Since particles do not leave the annular gap between the spherical shells, their total volume V_s, say, (assumed given) is a prescribed constant, independent of the flow conditions. Hence, the following global condition imposed upon ϕ must be satisfied:

$$\int_{\widehat{R}_1}^{\widehat{R}_2} 4\pi\hat{r}^2\phi(\hat{r})\,\mathrm{d}\hat{r} = \frac{V_s}{a^3}, \tag{7}$$

where $\widehat{R}_i = R_i/a$ $(i = 1, 2)$.

In combination, Eqs. (6) and (7) render unique the solution for $\phi(\hat{r})$ within the annular domain. Numerical solution of (6), following the assumption of an arbitrary value of ϕ at $\widehat{R}_1$ (unknown a priori), is straightforward. By such means, a trial-and-error procedure was used until (7) was satisfied. Computations were performed until successive values of ϕ differed by no more than 0.1%. Figs. 1a and b, and 2a and b address the respective cases of positive and negative flow rates for different values of V_s. (As will be shown a posteriori, the explicit value chosen for V_s has no effect upon the validity of our general conclusions below.)

For the case $Q > 0$, Fig. 1a and b, illustrate the resulting particle distributions for both small and large values of $\widehat{R}_1$, namely 5 and 100 particle radii, respectively. These figures reveal that,

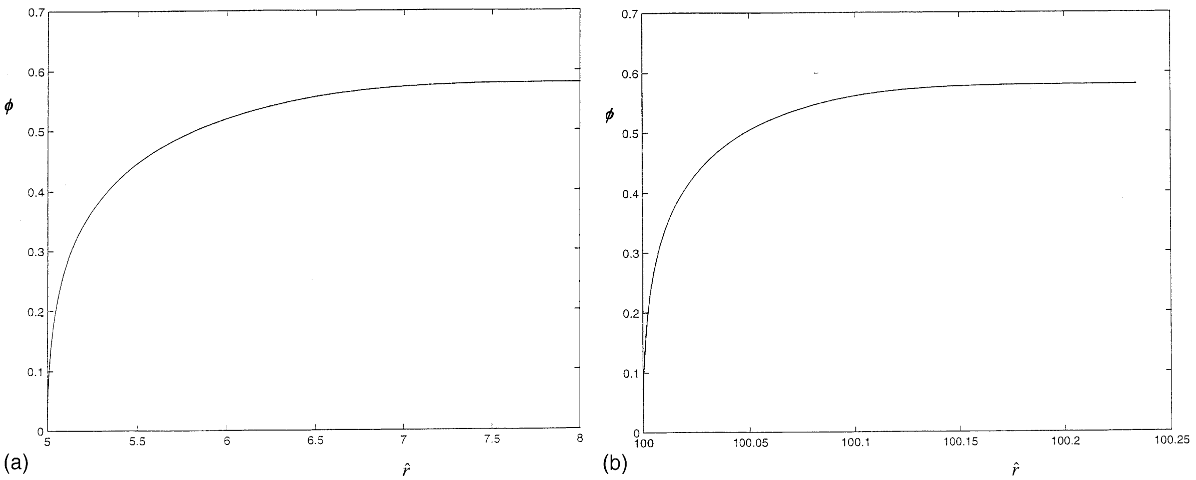

Fig. 1. (a) The volumetric particle concentration distribution ϕ for positive flow rates ($Q > 0$), and for an inner wall radius (made dimensionless with the suspended particle radius) of $\widehat{R}_1 = 5$. A rapid increase of concentration occurs within a single particle radius, leveling off at an asymptotic value of $\phi^* = 0.58$, the maximum particulate fraction beyond which the suspension can no longer flow. The particle concentration gradient is positive over the whole $\hat{r}$-range. (b) The volumetric particle concentration distribution ϕ for positive flow rates ($Q > 0$), and for an inner wall radius (made dimensionless with the suspended particle radius) of $\widehat{R}_1 = 100$. A rapid increase of concentration occurs within a single particle radius, leveling off at an asymptotic value of $\phi^* = 0.58$, the maximum particulate fraction beyond which the suspension can no longer flow. A sharp boundary exists between the suspension and the clear supernatant fluid. The particle concentration gradient is positive over the whole $\hat{r}$-range.

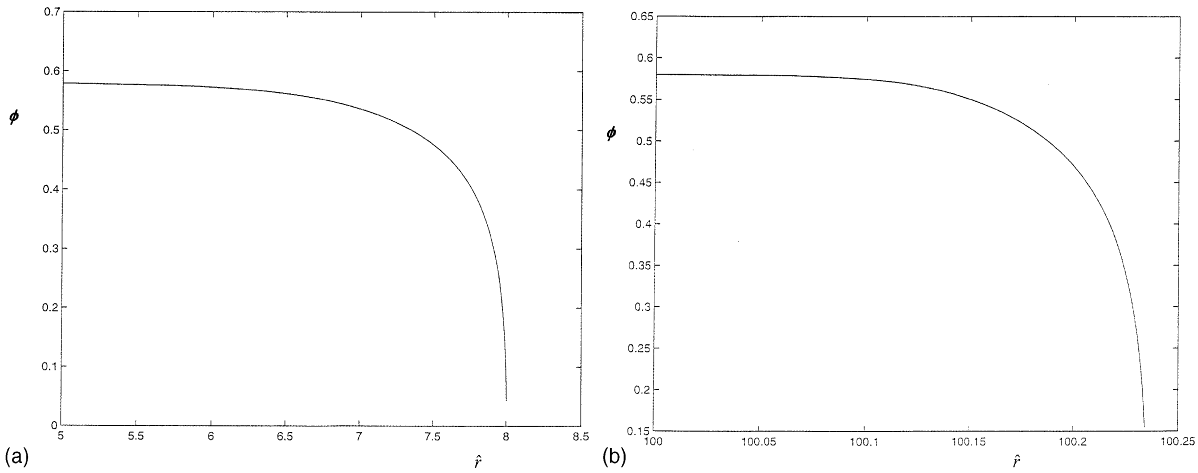

Fig. 2. (a) The volumetric particle concentration distribution ϕ for negative flow rates ($Q < 0$), and for an inner wall radius (made dimensionless with the suspended particle radius) of $\widehat{R}_1 = 5$. A rapid decrease of concentration occurs within a single particle radius. The radial particle concentration gradient is negative over the entire $\hat{r}$-range. (b) The volumetric particle concentration distribution ϕ for negative flow rates ($Q < 0$), and for an inner wall radius (made dimensionless with the suspended particle radius) of $\widehat{R}_1 = 100$. A rapid decrease of concentration occurs within a single particle radius. A sharp boundary exists between the suspension and the clear supernatant fluid. The particle concentration gradient is negative over the entire $\hat{r}$-range.

beginning at the inner sphere surface, the particle concentration at first experiences a very sharp increase with radial distance over a distance of the order of a particle radius. Beyond that point the particle concentration levels off, asymptotically attaining the maximum concentration of $\phi^* = 0.58$. Thus, for $\widehat{R}_1$ values larger than 100, essentially all of the particles congregate near the outer wall, creating a relatively sharp boundary between the suspension layer and the clear fluid. As such, it closely mimics a porous medium. The boundary location relative to the outer wall is simply determined by the total volume V_s occupied by the trapped particles. In addition, Fig. 1a and b reveal the radial particle concentration gradient, $d\phi/d\hat{r}$ to be positive over the entire $\hat{r}$-range. Similar conclusions can be drawn from Fig. 2a and b, which address the converse case of negative flow rates. Here, however, particles tend to congregate near the inner wall, while the radial particle concentration gradient $d\phi/d\hat{r}$ is negative over the entire $\hat{r}$-range.

Knowledge of the particle distribution now enables Eq. (1a) to be solved for the pressure gradient. Towards this end, introduce Eq. (3) into (1a) to obtain

$$\frac{dp}{dr} = -\frac{d\mu}{d\phi}\frac{Q}{\pi r^3}\frac{d\phi}{dr}, \tag{8}$$

since $\nabla^2\mathbf{v} = \mathbf{0}$ for radially symmetric flows. For $Q > 0$ we found that $d\phi/d\hat{r} > 0$ for all $\hat{r}$. Hence, since $d\mu/d\phi > 0$, it follows that $dp/d\hat{r} < 0$—a plausible result, since a diminution in pressure in the direction of flow is obviously required to maintain the flow. However, in the converse radial inflow case, namely $Q < 0$, we found that $d\phi/d\hat{r} < 0$, again requiring that $dp/d\hat{r} < 0$ according to Eq. (8). This implies that the pressure *increases* in the direction of flow, in clear contradiction to Darcy's law for interstitial fluid motion relative to a fixed bed of particles! In addition, the local pressure gradient depends upon the suspension viscosity gradient, whereas were Darcy's law assumed to apply in present circumstances, the local pressure gradient should depend (linearly) only upon the insterstitial viscosity itself, not the suspension-scale viscosity gradient!

3. Final remarks

Several lessons can be gleaned from the simple example outlined above. Despite the plausible concentration distribution predicted by the theory, it is questionable whether such distributions of particulate matter can be derived entirely from kinematical considerations alone (recall that this distribution was derived solely from Eqs. (1b) and (1c))—independently of the suspension-scale dynamics as embodied in the momentum Eq. (1a). In particular, when the suspension is closely packed, interparticle forces must affect the particle distribution in a fashion similar to that of an elastic porous medium subjected to a pressure gradient (see, for example the extensive review by Verruijt (1969)). In this context it is prudent to exercise caution when attempting to apply the system of Eqs. (1a)–(1c) significantly beyond their original scope (Leighton and Acrivos, 1987). In extending the theory to such cases one may suggest, for instance, that the diffusive flux needs to incorporate pressure gradient effects, much as such effects arise in the constitutive equation for the diffusion flux in the case of molecular transport processes (Bird et al., 1960; de Groot and Mazur, 1962).

In addition, we recognize that the dense 'deposit' formed by the suspended particles being filtered at the appropriate permeable spherical surface appears intuitively to be indistinguishable from a porous medium of the type encountered in classical slurry filtration theory, except perhaps for the fact that the porosity of the resulting porous medium is non-uniform (Tiller and Khatib, 1984; Tiller and Yeh, 1985; Tiller et al., 1996; Tiller and Kwon, 1998). Owing to this porous medium analogy, it is natural to expect the resulting flow to obey Darcy's law locally at any point within the 'filter cake', with the local permeability determined by the local porosity. In contrast, Eq. (8) indicates that the present momentum Eq. (1a) will predict local pressure gradients for *any* diffusional model that furnishes negative concentration gradients for negative flow rates. Since the existence of such negative gradients is an entirely plausible physical prediction for the case of radial inflow, this suggests that the momentum equation must be modified from its present form. In this context we may suggest (without proof) that the linear momentum Eq. (1a) needs to incorporate a stress contribution arising from the presence of strong particle concentration gradients, so that a Darcy-flow pressure gradient is attained at maximum packing (presumably the state achieved at high rates |Q| of inflow). A modification of the Navier–Stokes equation along these lines was already suggested by Korteweg (1901), albeit for the case of two miscible fluids of different densities, a restriction later modified by Joseph and Renardy (1993) to include species concentration gradients in place of mass-density gradients. Similar extensions of the momentum equation are also suggested by Ungarish (1993). Mechanics of mixtures may also provide guidelines of how to construct a comprehensive theory for a two phase fluid. Much of the theoretical progress in the mechanics of mixtures up to the mid-1980 can be found in the appendices to the book by Truesdell (1984) and the review articles by Bowen (1976), Atkin and Craine (1976) and Bedford and Drumheller (1983). More recent discussions can be found in the treatises by Rajagopal and Tao (1995) and Drew and Passman (1998).

Professor Andreas Acrivos of the City University of New York has raised an interesting counter-argument upon reviewing, at our request, the foregoing criticism of the diffusion model of lateral migration in terms of its apparent conflict with Darcy's law. His interpretation of the results for our case can be summarized by the following example: Assume that viscous fluid occupies the space between two concentric balloons of radii R_1 and R_2 Let Π_1, be the pressure within the small balloon, and Π_2 the pressure outside the big balloon. The resulting pressure difference $\Pi_1 - \Pi_2$ creates a radial flow (3). A normal stress balance across the surface $r = R_1$ yields $\tau_{rr}(R_1) + \Pi_1 = 0$ (assuming that the balloon possesses no hoop stress). A similar balance applies across the surface $r = R_1$. Upon combining these one obtains $\Pi_1 - \Pi_2 = \tau_{rr}(R_2) - \tau_{rr}(R_1)$. However, from (3), for a Newtonian suspension

$$\frac{d\tau_{rr}}{dr} = \frac{3\mu}{\pi r^4} Q. \tag{9}$$

Thus, for $Q > 0$ this yields the normal stress difference inequality $\tau_{rr}(R_2) - \tau_{rr}(R_1) > 0$, whence the *global pressure difference* $\Pi_1 - \Pi_2$ is positive. The converse is obviously true for $Q < 0$. In essence, Acrivos's claim is that in circumstances where the porosity of the porous medium varies, the direction in which the fluid flows depends upon the normal stress gradient, and that this direction may differ, depending, inter alia, upon whether the radial velocity increases or decreases with r.

Notwithstanding, Darcy's law, as it is currently understood, states specifically that the Darcy-scale stress system in a porous medium is isotropic, and that the *local* flow direction is dominated by the *local pressure* gradient (rather than by the local normal stress gradient, as is otherwise suggested by (9)). Non-isotropy of the suspension-scale stress system, essential in the derivation of

(9), is customarily considered to be of less significance in porous medium flows. Rather, anisotropy is normally introduced via Brinkman's equation (Brinkman, 1947) in an attempt to incorporate velocity-gradient corrections into Darcy's law (albeit only second-order corrections, presumably being no comparable to the first-order isotropic stresses).

Moreover, in limiting asymptotic circumstances for which the strong inequality $(R_2 - R_1)/R_1 \ll 1$ obtains, the radially symmetric flow field becomes formally equipollent with that for a homogeneous unidirectional flow between permeable walls. Moreover, the previously important directional distinction between 'inflow' and 'outflow' disappears in this limit. Application of the *unsteady*-state diffusion model Eq. (1c) to this configuration reveals that the particles, supposed initially uniformly distributed between the walls, will migrate towards the downstream wall. Eventually, when a steady state is attained, two distinct regions emerge—an upstream region composed of the pure interstitial fluid, and a downstream region consisting of a suspension characterized by a uniform volume fraction, one approaching ϕ^*, and where suspended particles are effectively in 'contact' with the downstream wall. According to Eq. (1a), within both regions the normal stress gradient vanishes (as too does the pressure gradient) regardless of the depth of the 'deposited' suspension layer on this wall. This conclusion contrasts starkly with what one would expect based upon a Darcy- or Darcy–Brinkman law analysis of the one-dimensional flow case. Moreover, based upon the conception of the transport process as a Navier–Stokes flow characterized by an inhomogeneous (suspension) viscosity, the steady-state solution predicts that no energy will be dissipated in either of the two regions. That no work is required to drive the interstitial fluid through the stagnant suspension is surely an aphysical prediction, again pointing up a deficiency of the model!

In summary, if Acrivos's interpretation is indeed tenable, it behooves one to seriously reexamine the general validity of Darcy's law in circumstances where gradients exist in both the local porosity and seepage velocity. In particular, issues arise immediately in even the most elementary context involving flow through porous media during the act of forming the divergence of the Darcy seepage velocity constitutive equation, $\mathbf{v} = -(k/\mu_0)\nabla p$. For in circumstances where the porosity varies, this introduces the Darcy permeability gradient ∇k into the basic transport equation, owing to the functional dependence of permeability upon porosity. However, we are unaware of any comparable discussion of the incorporation of such gradient effects into the corresponding viscosity coefficient appearing in Darcy's law. The obvious reason for this is that the pertinent Darcy viscosity coefficient is invariably taken to be the interstitial fluid viscosity μ_0 rather than the suspension viscosity μ, although the two choices are far from being equivalent in their physical consequences when porosity gradients exist. Similar questions may arise in regard to the proper Brinkman viscosity coefficient μ' appearing in a modified Darcy–Brinkman equation $\nabla p = \mu'\nabla^2\mathbf{v} + 2(\nabla\mu')\cdot\mathbf{v} - (\mu_0/k)\mathbf{v}$ (one may suggest to modify the Brinkman equation by adding the second term on the r.h.s. to account for the variable viscosity field). Explicitly, is the conventional interstitial fluid viscosity coefficient, μ_0 appearing in the Darcy contribution $-(\mu_0/k)\mathbf{v}$ to the pressure gradient ∇p the same as the so-called Brinkman viscosity coefficient, μ', say, appearing in the comparable Brinkman contribution $\mu'\nabla^2\mathbf{v}$ to the pressure gradient? (For some theoretical and computational discussions of the Brinkman viscosity issue, see Freed and Muthukumar, 1978; Koplik et al., 1983; Kim and Russell, 1985; Larson and Higdon, 1986; Durlofsky and Brady, 1987; Chang and Acrivos, 1988; Martys et al., 1994; experimental data bearing on the subject, are presented by Beavers and Joseph (1967), Beavers et al. (1970), and

Givler and Altobelli (1994). The entire Brinkman viscosity issue was extensively reviewed several years ago by Batycky (1995)).

Given the largely empirical basis of these controversial issues, the fundamental question posed here is best left for the time being as a major unresolved question, one clearly requiring future investigation. From this perspective, our example has merely focused on the need for such fundamental studies.

Acknowledgements

This work was supported by the Office of Basic Energy Sciences of the U.S. Department of Energy as well as by a 'Grand Challenge' award sponsored by the Mathematical, Information, and Computational Sciences Division, Office of Computational and Technology Research, Office of Energy Research, U.S. Department of Energy. We thank Professor Acrivos for his comments on an earlier draft of this paper.

References

Acrivos, A., Mauri, R., Fan, X., 1993. Shear-induced resuspension in a Couette device. Int. J. Multiphase Flow 19, 797–802.

Atkin, R.J., Craine, R.E., 1976. Continuum theories of mixtures—basic theory and historical development. Q. J. Mech. Appl. Math. 29, 209–244.

Batycky, R.P., 1995. Inhomogeneous Stokes flow through porous media. Ph.D. Thesis, Massachusetts Institute of Technology, 273 pp.

Beavers, G.S., Joseph, D.D., 1967. Boundary conditions at a naturally permeable wall. J. Fluid Mech. 30, 197–207.

Beavers, G.S., Sparrow, E.M., Magnuson, R.A., 1970. Experiments on parallel flows in a channel and a bounding porous medium. J. Basic Eng. 92, 843–848.

Bird, R.B., Stewart, W.E., Lightfoot, E.N., 1960. Transport Phenomena. Wiley, New York.

Bedford, A., Drumheller, D.S., 1983. Theories of immiscible and structured mixtures. Int. J. Eng. Sci. 21, 863–960.

Bossis, G., Brady, J.F., 1988. Dynamic simulation of sheared suspensions. I. General method. J. Chem. Phys. 80, 5141–5154.

Bowen, R.M., 1976. In: Part I, Theory of mixtures, Eringen, A.C. (Ed.), Continuum Physics, vol. III. Academic, New York.

Brinkman, H.C., 1947. A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. Appl. Sci. Res. A 1, 27–34.

Chang, E.Y., Acrivos, A., 1988. The effective permeability of a random dispersion of spheres. PhysicoChem. Hydrodyn. 10, 579–584.

de Groot, S.R., Mazur, P., 1962. Non-Equilibrium Thermodynamics. North-Holland, Amsterdam.

Drew, D.A., Passman, S.L., 1998. Theory of Multicomponent Fluids. Springer-Verlag, New York.

Durlofsky, L., Brady, J.F., 1987. Analysis of the Brinkman equation as a model for flow in porous media. Phys. Fluids 30, 3329–3341.

Freed, K.F., Muthukumar, M., 1978. On the Stokes problem for a suspension of spheres at finite concentrations. J. Chem. Phys. 68, 2088–2096.

Givler, R.C., Altobelli, S.A., 1994. A determination of the effective viscosity for the Brinkman–Forcheimer model. J. Fluid Mech. 258, 355–370.

Joseph, D.D., Renardy, Y.Y., 1993. Fundamentals of Two-Fluid Dynamics. Part II: Lubricated Transport, Drops and Miscible Liquids. Springer-Verlag, New York.

Kim, S., Russell, W.B., 1985. Modeling of porous media by renormalization of the Stokes equations. J. Fluid Mech. 154, 269–286.

Koh, C.J., Hookham, P.A., Leal, L.G., 1994. An experimental investigation of concentrated suspension flows in a rectangular channel. J. Fluid Mech. 266, 1–32.

Koplik, J., Levin, H., Zee, 1983. Viscosity renormalization in the Brinkman equation. Phys. Fluids 26, 2864–2870.

Korteweg, D., 1901. Sur la forme que prennent les equations du mouvement des fluids si I'on tient compte des forces capillaires causees par des variations de densite. Arch. Neerl. Sci. Exactes et Naturelles Series II 6, 1–9.

Krieger, I.M., 1972. Rheology of monodispersed lattices. Adv. Colloid Interf. Sci. 3, 111–136.

Larson, R.E., Higdon, J.J.L., 1986. Microscopic flow near the surface of two-dimensional porous media. Part 1. Axial flow. J. Fluid Mech. 166, 449–472.

Leighton, D., Acrivos, A., 1986. Viscous resuspension. Chem. Eng. Sci. 41, 1377–1384.

Leighton, D., Acrivos, A., 1987. The shear induced migration of particles in concentrated suspensions. J. Fluid Mech. 181, 415–439.

Martys, N., Bentz, D.P., Garboczi, E.J., 1994. Computer simulation study of the effective viscosity in Brinkman's equation. Phys. Fluids 6, 1434–1439.

Miskin, L., Elliot, L., Ingham, D.B., 1999. Stability in a fully developed two-dimensional resuspension flow. Int. J. Multiphase Flow 25, 501–530.

Nir, A., Acrivos, A., 1990. Sedimentation and sediment flow on inclined surfaces. J. Fluid Mech. 212, 139–153.

Nott, P.R., Brady, J.F., 1994. Pressure-driven flow of suspensions: simulations and theory. J. Fluid Mech. 275, 157–199.

Philips, R.J., Armstrong, R.C., Brown, R.A., 1992. A constitutive equation for concentrated suspensions that accounts for shear-induced particle migration. Phys. Fluids A 4, 30–40.

Rajagopal, K.R., Tao, L., 1995. Mechanics of Mixtures. World Scientific, Singapore.

Schaflinger, U., Acrivos, A., Zhang, K., 1990. Viscous resuspension of sediment within a laminar and stratified flow. Int. J. Multiphase Flow 16, 576–578.

Tiller, F.M., Khatib, Z., 1984. The theory of sediment volumes of compressible particulate structures. J. Colloid Interf. Sci. 100, 55–67.

Tiller, F.M., Yeh, C.S., 1985. The role of porosity in filtration: 10. Deposition of compressible cakes on external radial surfaces. AIChE J. 31, 1241–1248.

Tiller, F.M., Kirby, J.M., Nguyen, H.L., 1996. Approximate theory for radial filtration/consolidation. J. Geotech. Eng.—ASCE 122, 797–805.

Tiller, F.M., Kwon, J.H., 1998. The role of porosity in filtration: XIII. Behavior of highly compactible cakes. AIChE J. 44, 2159–2167.

Truesdell, C., 1984. Rational Thermodynamics. Springer-Verlag, New York.

Ungarish, M., 1993. Hydrodynamics of Suspension Flows: Fundamentals of Centrifugal and Gravity Separation. Springer-Verlag, Berlin.

Verruijt, A., 1969. Elastic storage of aquifers. In: De Wiest, R.J.M. (Ed.), Flow Through Porous Media. Academic, New York.

Zhang, K., Acrivos, A., 1994. Viscous resuspension in fully developed laminar pipe flows. Int. J. Multiphase Flow 20, 579–591.